

Forbenius method

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$x_0 = 0$$

O.P (ordinary point)

$$P(0) \neq \infty$$

$$Q(0) \neq \infty$$

$$y = \sum a_n x^n$$

وتقارب أقل أس

والتي بعده والعلم

(S.P) Singular Points

$$P(0) = \infty \text{ or } Q(0) = \infty$$

I.R Regular singular Point (I.S.P)

$$x P(x) \Big|_{x=0} = \infty$$

or

$$x^2 Q(x) \Big|_{x=0} = \infty$$

ليس لها حل بالمسلسلة

Regular singular Point (R.S.P)

$$x P(x) \Big|_{x=0} \neq \infty$$

and

$$x^2 Q(x) \Big|_{x=0} \neq \infty$$

* نفرض الحل :-

$$y = \sum a_n x^{n+\lambda}$$

نعوض في المعادلة ونقارن معامل أقل أس ينتج معادلة في λ .

$$a\lambda^2 + b\lambda + c = 0$$

شكل الجذور

↓ [1]

جذور غير متساوية

$$\lambda_1 \neq \lambda_2$$

$$(\lambda_1 - \lambda_2) \text{ is } \text{non integer} \quad y_1 = \sum a_n x^{n+\lambda}$$

non integer

$$y_1 = \sum a_n x^{n+\lambda_1}$$

$$y_2 = \sum a_n x^{n+\lambda_2}$$

$$y_{G.S} = c_1 y_1 + c_2 y_2$$

↓ [2]

$$\lambda_1 = \lambda_2 = \lambda$$

$$y_1 = \sum a_n x^{n+\lambda}$$

$$y_2 = \left. \frac{\partial y(x, \lambda)}{\partial \lambda} \right|_{\lambda=\lambda_1}$$

$$y_{G.S} = c_1 y_1 + c_2 y_2$$

$$y_2 = y_1(x) \ln x + \left. \sum \frac{\partial a_n}{\partial \lambda} x^{n+\lambda} \right|_{\lambda=\lambda_1}$$

نحسب a_n في كل خطوة

وبدأنا (2) ثم نفاضل

الناتج ونعرف في الخطوة
في الخطوة التالية

Notes

$$y(x, \lambda) = \sum a_n(\lambda) x^{n+\lambda}$$

$$\left. \frac{\partial y}{\partial \lambda} \right|_{\lambda=\lambda_1} = \sum a_n x^{n+\lambda} (1) \ln x \Big|_{\lambda=\lambda_1} + \sum \frac{\partial a_n}{\partial \lambda} x^{n+\lambda} \Big|_{\lambda=\lambda_1}$$

$$+ \sum \frac{\partial a_n}{\partial \lambda} x^{n+\lambda} \Big|_{\lambda=\lambda_1}$$

[2] Lec 20

$$\boxed{3} \quad \lambda_1 - \lambda_2 = \text{integer} \quad \text{الفرد عدد صحيح}$$

$$y_1 = \sum a_n x^{n+\lambda_1}$$

$$y_2 = \frac{\partial}{\partial \lambda} [(\lambda - \lambda_2) y(x, \lambda)] \Big|_{\lambda = \lambda_2}$$

$$y_{G.S} = C_1 y_1 + C_2 y_2$$

→ نحسب y و y' عند x_0 ونعوض الناتج في $(\lambda - \lambda_2)$ ونقارن.

$$\boxed{\text{Ex}} \text{ solve : } 4x y'' + 2y' + y = 0 \quad \text{near } x_0 = 0$$

$\boxed{\text{Sol}}$

$$y'' + P(x)y' + Q(x)y = F(x)$$

$$y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0$$

$$P(x) = \frac{1}{2x} \quad ; \quad Q(x) = \frac{1}{4x}$$

$$P(0) = \infty$$

$$Q(0) = \infty \Rightarrow (S.P)$$

$$x P(x) \Big|_{x=0} = x \left(\frac{1}{2x} \right) \Big|_{x=0} = \frac{1}{2} \neq \infty$$

$\boxed{3} \text{ Lec 2.}$

$$x^2 Q(x) \Big|_{x=0} = x^2 \left(\frac{1}{4x} \right) \Big|_{x=0} = 0 \neq \infty$$

\Rightarrow is R.S.P.

$$y = \sum a_n x^{n+\lambda}$$

$$\dot{y} = \sum (n+\lambda) a_n x^{n+\lambda-1}$$

$$\ddot{y} = \sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-2}$$

$$4 \sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-1} + 2 \sum (n+\lambda) a_n x^{n+\lambda-1} + \sum a_n x^{n+\lambda} = 0$$

← لمعرفة أقل أس نضع $(n=0)$ في كل الحدود ونقارن معامل أشهر أس

← نضع $(n=0)$ في ~~جميع~~ المتسلسلات ونوجد أقل أس ومعامل

فيكون يعطى المعادلة المميزة في λ .

Co-eff of $x^{\lambda-1}$

$$4\lambda(\lambda-1)a_0 + 2\lambda a_0 + \dots = 0 \text{ لا يوجد}$$

$$\lambda a_0 (2\lambda - 2 + 1) = 0$$

$$\downarrow \\ \neq 0$$

$$\lambda_1 = 0 \quad \text{or} \quad \lambda_2 = \frac{1}{2}$$

← جذور غير متساوية والفرق يساوي عدد غير صحيح.

$$y_1 = \sum a_n x^{n+\lambda_1} \quad ; \quad y_2 = \sum a_n x^{n+\lambda_2}$$

Coeff. x^λ

$$4(\lambda+1)\lambda a_1 + 2(\lambda+1)a_1 + a_0 = 0$$

$$2(\lambda+1)a_1 [2\lambda+1] = -a_0$$

$$a_1 = \frac{-a_0}{2(\lambda+1)(2\lambda+1)}$$

Coeff. of $x^{n+\lambda}$

$$4(n+\lambda+1)(n+\lambda)a_{n+1} + 2(n+\lambda+1)a_{n+1} + a_n = 0$$

$$2(n+\lambda+1)a_{n+1} [2n+2\lambda+1] = -a_n$$

$$a_{n+1} = \frac{-a_n}{2(n+\lambda+1)(2n+2\lambda+1)}$$

• Put $n=1$

$$a_2 = \frac{-a_1}{2(\lambda+2)(2\lambda+3)} = \frac{a_0}{2(\lambda+2)(2\lambda+3)2(\lambda+1) * (2\lambda+1)}$$

مضروب في القواسم

Case (1) $\lambda_1 = 0$

$$y_1 = \sum a_n x^{n+\lambda_1} = \sum a_n x^n x^{\lambda_1}$$

$$= x^{\lambda_1} (a_0 + a_1 x + a_2 x^2 \dots) \Big|_{\lambda=0}$$

$$y_1 = a_0 - \frac{a_0}{2} x + \frac{a_0}{(2)(3)(2)(2)} x^2 \dots$$

$$y_1 = a_0 \left[1 - \frac{x}{2} + \frac{x^2}{24} \dots \right]$$

Case (2) $\lambda_2 = \frac{1}{2}$

$$y_2 = \sum a_n x^{n+\lambda_2} = \sum a_n x^n x^{\lambda_2}$$

$$= x^{\frac{1}{2}} [a_0 + a_1 x + a_2 x^2 \dots]$$

$$= x^{\frac{1}{2}} \left[a_0 - \frac{a_0}{(2) \left(\frac{3}{2}\right)(2)} x + \frac{a_0}{2 \left(\frac{5}{2}\right)(4) \left(\frac{3}{2}\right)(2)} x^2 - \dots \right]$$

$$y_{G.S} = C_1 y_1 + C_2 y_2$$

Ex solve by series f_n .

$$x^2 \ddot{y} - x \dot{y} + (x^2 + 1)y = 0 \quad \text{at } x_0 = 0$$

Sol

$$\ddot{y} + P(x) \dot{y} + Q(x) y = f(x)$$

$$\ddot{y} - \frac{1}{x} \dot{y} + \frac{x^2 + 1}{x^2} y = 0$$

$$P(x) = \frac{-1}{x} \quad ; \quad Q(x) = \frac{x^2 + 1}{x^2}$$

$$P(0) = \infty \quad ; \quad Q(0) = \infty \quad \text{is S.P}$$

$$x P(x) \Big|_{x=0} = x \left(\frac{-1}{x} \right) \Big|_{x=0} = -1 \neq \infty$$

$$x^2 Q(x) \Big|_{x=0} = x^2 \left(\frac{-1}{x} \right) \Big|_{x=0} \neq \infty$$

\Rightarrow R.S.P

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$$y = \sum a_n x^{n+\lambda}$$

$$\dot{y} = \sum a_n (n+\lambda) x^{n+\lambda-1}$$

$$y'' = \sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-2}$$

$$\sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda} - \sum a_n (n+\lambda) x^{n+\lambda}$$

$$+ \sum a_n x^{n+\lambda+2} + \sum a_n x^{n+\lambda} = 0$$

• اقل أس هو x^λ

* Coeff. of x^λ

$$\cancel{\lambda(\lambda-1)a_0} - \cancel{\lambda a_0} + \cancel{a_0} = 0$$

$$\lambda(\lambda-1)a_0 - \lambda a_0 + a_0 = 0$$

$$a_0 [\lambda^2 - 2\lambda + 1] = 0$$

\downarrow
 $\neq 0$

$$(\lambda-1)(\lambda-1) = 0$$

$$\lambda_1 = 1, \lambda_2 = 1$$

جذور متساوية

$$y_1 = \sum a_n x^{n+\lambda_1}$$

$$y_2 = \int y_1 \ln x + \sum \left. \frac{\partial a_n}{\partial \lambda} \right|_{\lambda=\lambda_1} x^{n+\lambda_1}$$

* Coeff of $x^{\lambda+1}$

$$(\lambda+1)\lambda a_1 - (\lambda+1)a_1 + \text{وجود} + a_1 = 0$$

$$a_1 [(\lambda+1)(\lambda+1) + 1] = 0 \Rightarrow a_1 = 0$$

معناها 1 = $\lambda_2 \neq \lambda_1$ لأنه ما يدخل القوس لا يساوي صفر.

* Coeff. of $x^{n+\lambda}$

$$(n+\lambda)(n+\lambda-1)a_n - (n+\lambda)a_n + a_{n-2} + a_n = 0$$

$$a_n \left[\underbrace{(n+\lambda)(n+\lambda-1-1)}_{(n+\lambda)-2} + 1 \right] = -a_{n-2}$$

$$a_n = \frac{-a_{n-2}}{(n+\lambda)^2 - 2(n+\lambda) + 1}$$

$$a_n = \frac{-a_{n-2}}{(n+\lambda-1)^2}$$

$$n=2$$

$$a_2 = -\frac{a_0}{(\lambda+1)^2}$$

$$\underline{n=3}$$

$$a_3 = \frac{-a_1}{(\lambda+2)^2} = 0$$

$$\underline{\text{Case (1)}} \quad \lambda_1 = 1 ; y_1 = \sum a_n x^{n+\lambda_1}$$

$$y_1 = x^{\lambda_1} \sum a_n x^n \Big|_{\lambda=\lambda_1} = x \left[a_0 + a_1 x + a_2 x^2 + \dots \right] \Big|_{\lambda=1}$$

$$= x \left[a_0 + 0 + \frac{a_0}{4} x^2 + \dots \right]$$

$$\underline{\text{Case (2)}} \quad \lambda_2 = 1 ; y_2 = y_1 \ln x + \sum \frac{\partial a_n}{\partial \lambda} x^{n+\lambda_1} \Big|_{\lambda=\lambda_2}$$

$$y_2 = y_1 \ln x + x^{\lambda_2} \left[\frac{\partial a_0}{\partial \lambda} + \frac{\partial a_1}{\partial \lambda} x + \frac{\partial a_2}{\partial \lambda} x^2 + \dots \right]$$

$$a_1 = 0 \Rightarrow \frac{\partial a_1}{\partial \lambda} = 0$$

$$a_2 = \frac{-a_0}{(\lambda+1)^2} = -a_0 (\lambda+1)^{-2}$$

$$\frac{\partial a_2}{\partial \lambda} = +2 (\lambda+1)^{-3} a_0$$

← تعامل مع a_0 كأنه ثابت

ولا تنوع تفاضلياً بالنسبة لـ λ

تساوى صفر.

$$a_0^* = \frac{\partial a_0}{\partial \lambda}$$

← ممكن نكتبها a_0^*

$$a_2 = 2 (\lambda+1)^{-3} a_0 - (\lambda+1)^{-2} a_0^* \Big|_{\lambda=1}$$

$$a_2 = \frac{2a_0}{8} - \frac{a_0^*}{4}$$

بالعكس في y_2

$$y_2 = y_1 \ln x + x \left[a_0^* + 0 + \left(\frac{2a_0}{8} - \frac{a_0^*}{4} \right) x^2 + \dots \right]$$

$$y_{G.S} = c_1 y_1 + c_2 y_2$$

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Homework

Solve ~~the~~ by series for

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

near $x_0 = 0$

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